Estudio de las ecuaciones

entre

derivadas parciales de cuarto orden

con dos variables independientes

1904

Los trabajos realizados por algunos matemáticos notables, y en particular por M. Guldberg, de Christiania, sobre ecuaciones entre derivadas parciales de tercer orden, nos han impulsado a extender dicho estudio a las ecuaciones de cuarto orden.

Antes de entrar en materia, bueno será, para mayor inteligencia del lector, indicar la marcha general que debe seguirse para alcanzar una primera integral de la ecuación dada.

Cuanto antes interesa determinar el sistema de ecuaciones entre derivadas parciales, que debe tener lugar para que se realice la condición de que la ecuación dada admita una primera integral; una vez hallado este sistema, conviene transformarlo en otro que sea lineal y de primer orden, y por último, mediante coeficientes indeterminados, reducir el sistema precedente a una sola ecuación lineal entre derivadas parciales de primer orden, en cuyo caso, se pasa fácilmente a un sistema de ecuaciones diferenciales totales, que deben procurar definitivamente dos integrales particulares, las cuales, puestas una en función de la otra, dan por fin la integral pedida de la ecuación dada.

Dos casos pueden presentarse en el presente estudio, según se considere o no la ecuación lineal y completa respecto a $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1$; y conforme a las notaciones siguientes:

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q;$$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x\,\partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t;$$

$$\frac{\partial^3 z}{\partial x^3} = \alpha, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = \beta, \quad \frac{\partial^3 z}{\partial x\,\partial y^2} = \gamma, \quad \frac{\partial^3 z}{\partial y^3} = \delta;$$

$$\frac{\partial^4 z}{\partial x^4} = \alpha_1, \quad \frac{\partial^4 z}{\partial x^3 \partial y} = \beta_1, \quad \frac{\partial^4 z}{\partial x^2\,\partial y^2} = \gamma_1, \quad \frac{\partial^4 z}{\partial x\,\partial y^3} = \delta_1, \quad \frac{\partial^4 z}{\partial y^4} = \varepsilon_1$$

Sea la ecuación:

$$A\alpha_1 + B\beta_1 + C\gamma_1 + D\delta_1 + E\varepsilon_1 + P = 0 \quad (1)$$

en el supuesto de que *A, B, C, D, E, P,* sean funciones de: $x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$.

En este concepto se puede afirmar que una ecuación entre derivadas parciales de tercer orden, tal como $u = f(v)$, siendo f una función arbitraria, puede considerarse integral de (1), si u y v, satisfacen a las condiciones que a continuación se expresan:

$$\frac{\partial(u,v)}{\partial(\beta,\alpha)} = 0, \quad \frac{\partial(u,v)}{\partial(\gamma,\alpha)} = 0, \quad \frac{\partial(u,v)}{\partial(\delta,\alpha)} = 0,$$
$$\frac{\partial(u,v)}{\partial(\gamma,\beta)} = 0, \quad \frac{\partial(u,v)}{\partial(\delta,\beta)} = 0, \quad \frac{\partial(u,v)}{\partial(\delta,\gamma)} = 0,$$

En efecto, de $u = f(v)$, al diferenciar, se tiene $du = f'(v)dv$ y como f es una función arbitraria, debe resultar $du = 0, dv = 0$. Así, pues, se tiene:

$$(2) \quad \begin{cases} \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq + \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial \alpha}d\alpha + \frac{\partial u}{\partial \beta}d\beta + \frac{\partial u}{\partial \gamma}d\gamma + \frac{\partial u}{\partial \delta}d\delta = 0 \\ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz + \frac{\partial v}{\partial p}dp + \frac{\partial v}{\partial q}dq + \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial s}ds + \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial \alpha}d\alpha + \frac{\partial v}{\partial \beta}d\beta + \frac{\partial v}{\partial \gamma}d\gamma + \frac{\partial v}{\partial \delta}d\delta = 0 \end{cases}$$

Además se sabe que:

$$(2') \quad \begin{cases} dz = pdx + qdy, & dp = rdx + \varepsilon dy, & dq = \varepsilon dx + tdy \\ dr = \alpha dx + \beta dy, & ds = \beta dx + \gamma dy, & dt = \gamma dx + \delta dy, \\ d\alpha = \alpha_1 dx + \beta_1 dy, & & d\beta = \beta_1 dx + \gamma_1 dy \\ d\gamma = \gamma_1 dx + \delta_1 dy, & & d\delta = \delta_1 dx + \varepsilon_1 dy \end{cases}$$

Al sustituir estos valores en (2), ordenando según dx y dy, se obtiene:

$$(3) \quad \begin{cases} \left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} + \alpha\frac{\partial u}{\partial r} + \beta\frac{\partial u}{\partial s} + \gamma\frac{\partial u}{\partial t} + \alpha_1\frac{\partial u}{\partial \alpha} + \beta_1\frac{\partial u}{\partial \beta} + \gamma_1\frac{\partial u}{\partial \gamma} + \delta_1\frac{\partial u}{\partial \delta}\right]dx = \\ = -\left[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} + \beta\frac{\partial u}{\partial r} + \gamma\frac{\partial u}{\partial s} + \delta\frac{\partial u}{\partial t} + \beta_1\frac{\partial u}{\partial \alpha} + \gamma_1\frac{\partial u}{\partial \beta} + \delta_1\frac{\partial u}{\partial \gamma} + \varepsilon_1\frac{\partial u}{\partial \delta}\right]dy, \\ \left[\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q} + \alpha\frac{\partial v}{\partial r} + \beta\frac{\partial v}{\partial s} + \gamma\frac{\partial v}{\partial t} + \alpha_1\frac{\partial v}{\partial \alpha} + \beta_1\frac{\partial v}{\partial \beta} + \gamma_1\frac{\partial v}{\partial \gamma} + \delta_1\frac{\partial v}{\partial \delta}\right]dx = \\ = -\left[\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q} + \beta\frac{\partial v}{\partial r} + \gamma\frac{\partial v}{\partial s} + \delta\frac{\partial v}{\partial t} + \beta_1\frac{\partial v}{\partial \alpha} + \gamma_1\frac{\partial v}{\partial \beta} + \delta_1\frac{\partial v}{\partial \gamma} + \varepsilon_1\frac{\partial v}{\partial \delta}\right]dy. \end{cases}$$

A fin de reducir los cálculos, supondremos:

$$
(3')\quad \begin{cases}
\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial u}{\partial x}+p\,\dfrac{\partial u}{\partial z}+r\,\dfrac{\partial u}{\partial p}+s\,\dfrac{\partial u}{\partial q}+\alpha\,\dfrac{\partial u}{\partial r}+\beta\,\dfrac{\partial u}{\partial s}+\gamma\,\dfrac{\partial u}{\partial t}\,, \\[2ex]
\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial u}{\partial y}+q\,\dfrac{\partial u}{\partial z}+s\,\dfrac{\partial u}{\partial p}+t\,\dfrac{\partial u}{\partial q}+\beta\,\dfrac{\partial u}{\partial r}+\gamma\,\dfrac{\partial u}{\partial s}+\delta\,\dfrac{\partial u}{\partial t}\,, \\[2ex]
\left(\dfrac{\partial v}{\partial x}\right)=\dfrac{\partial v}{\partial x}+p\,\dfrac{\partial v}{\partial z}+r\,\dfrac{\partial v}{\partial p}+s\,\dfrac{\partial v}{\partial q}+\alpha\,\dfrac{\partial v}{\partial r}+\beta\,\dfrac{\partial v}{\partial s}+\gamma\,\dfrac{\partial v}{\partial t}\,, \\[2ex]
\left(\dfrac{\partial v}{\partial y}\right)=\dfrac{\partial v}{\partial y}+q\,\dfrac{\partial v}{\partial z}+s\,\dfrac{\partial v}{\partial p}+t\,\dfrac{\partial v}{\partial q}+\beta\,\dfrac{\partial v}{\partial r}+\gamma\,\dfrac{\partial v}{\partial s}+\delta\,\dfrac{\partial v}{\partial t}\,.
\end{cases}
$$

Según esta notación, después de dividir las dos ecuaciones (3), con objeto de eliminar *dx*, *dy***,** se halla:

$$
\frac{\left(\dfrac{\partial u}{\partial x}\right)+\alpha_1\dfrac{\partial u}{\partial \alpha}+\beta_1\dfrac{\partial u}{\partial \beta}+\gamma_1\dfrac{\partial u}{\partial \gamma}+\delta_1\dfrac{\partial u}{\partial \delta}}{\left(\dfrac{\partial v}{\partial x}\right)+\alpha_1\dfrac{\partial v}{\partial \alpha}+\beta_1\dfrac{\partial v}{\partial \beta}+\gamma_1\dfrac{\partial v}{\partial \gamma}+\delta_1\dfrac{\partial v}{\partial \delta}}=\frac{\left(\dfrac{\partial u}{\partial y}\right)+\beta_1\dfrac{\partial u}{\partial \alpha}+\gamma_1\dfrac{\partial u}{\partial \beta}+\delta_1\dfrac{\partial u}{\partial \gamma}+\varepsilon_1\dfrac{\partial u}{\partial \delta}}{\left(\dfrac{\partial v}{\partial y}\right)+\beta_1\dfrac{\partial v}{\partial \alpha}+\gamma_1\dfrac{\partial v}{\partial \beta}+\delta_1\dfrac{\partial v}{\partial \gamma}+\varepsilon_1\dfrac{\partial v}{\partial \delta}};
$$

ecuación que puede expresarse bajo la forma siguiente:

$$
\begin{aligned}
&\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)\right]+\alpha_1\left[\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \alpha}\right]+\beta_1\left[\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \beta}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \beta}\right]\\
&+\gamma_1\left[\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \gamma}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \gamma}\right]+\delta_1\left[\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \delta}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \delta}\right]\\
&+\varepsilon_1\left[\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \delta}-\frac{\partial u}{\partial \delta}\left(\frac{\partial v}{\partial x}\right)\right]+\alpha_1\gamma_1\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \alpha}\right]+\alpha_1\delta_1\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \alpha}\right]+\alpha_1\varepsilon_1\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \delta}-\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \alpha}\right]\\
&+\beta_1^2\left[\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta}\right]+\beta_1\gamma_1\left[\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \gamma}\right]+\beta_1\delta_1\left[\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \beta}+\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \delta}\right]\\
&+\beta_1\varepsilon_1\left[\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \delta}-\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \beta}\right]+\gamma_1^2\left[\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \gamma}\right]+\gamma_1\delta_1\left[\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \delta}\right]+\gamma_1\varepsilon_1\left[\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \delta}-\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \gamma}\right]\\
&+\delta_1^2\left[\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \delta}\right]=0
\end{aligned}
$$

Y en el supuesto de que:

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)=P, \qquad \frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \alpha}=A,$$

$$\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \beta}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \beta}=B,$$

$$\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \gamma}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \gamma}=C,$$

$$\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial \delta}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \delta}=D,$$

$$\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \delta}-\frac{\partial u}{\partial \delta}\left(\frac{\partial v}{\partial x}\right)=E,$$

$$\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta}=\frac{\partial(u,v)}{\partial(\beta,\alpha)}=F, \qquad \frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \gamma}=\frac{\partial(u,v)}{\partial(\gamma,\alpha)}=G,$$

$$\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \alpha}-\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \delta}=\frac{\partial(u,v)}{\partial(\delta,\alpha)}=H, \qquad \frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \gamma}=\frac{\partial(u,v)}{\partial(\gamma,\beta)}=L,$$

$$\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \beta}-\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \delta}=\frac{\partial(u,v)}{\partial(\delta,\beta)}=M, \qquad \frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \gamma}-\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \delta}=\frac{\partial(u,v)}{\partial(\delta,\gamma)}=N,$$

se obtiene:

$$(4) \quad \begin{cases} A\alpha_1+B\beta_1+C\gamma_1+D\delta_1+E\varepsilon_1 \\ +F\left(\beta_1^2-\alpha_1\gamma_1\right)+G\left(\gamma_1\beta_1-\alpha_1\delta_1\right)+H\left(\delta_1\beta_1-\alpha_1\varepsilon_1\right) \\ +L\left(\gamma_1^2-\beta_1\delta_1\right)+M\left(\delta_1\gamma_1-\beta_1\varepsilon_1\right)+N\left(\delta_1^2-\gamma_1\varepsilon_1\right)+P=0 \end{cases}$$

Así pues, para que esta ecuación se transforme en (1), es preciso que $F = G = H = L = M = N = 0$, conforme nos habíamos propuesto demostrar.

Veamos ahora a que sistema de ecuaciones entre derivadas parciales deben satisfacer las condiciones precedentes. Estas condiciones permiten escribir las igualdades que a continuación se expresan:

$$\frac{\dfrac{\partial u}{\partial \beta}}{\dfrac{\partial u}{\partial \alpha}}=\frac{\dfrac{\partial v}{\partial \beta}}{\dfrac{\partial v}{\partial \alpha}}, \qquad \frac{\dfrac{\partial u}{\partial \gamma}}{\dfrac{\partial u}{\partial \alpha}}=\frac{\dfrac{\partial v}{\partial \gamma}}{\dfrac{\partial v}{\partial \alpha}}, \qquad \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial u}{\partial \alpha}}=\frac{\dfrac{\partial v}{\partial \delta}}{\dfrac{\partial v}{\partial \alpha}}$$

$$\frac{\dfrac{\partial u}{\partial \gamma}}{\dfrac{\partial u}{\partial \beta}}=\frac{\dfrac{\partial v}{\partial \gamma}}{\dfrac{\partial v}{\partial \beta}}, \qquad \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial u}{\partial \beta}}=\frac{\dfrac{\partial v}{\partial \delta}}{\dfrac{\partial v}{\partial \beta}}, \qquad \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial u}{\partial \gamma}}=\frac{\dfrac{\partial v}{\partial \delta}}{\dfrac{\partial v}{\partial \gamma}}.$$

Si designamos por ***m, n, l,*** los segundos miembros de las tres primeras igualdades, se tiene:

$$(4')\quad \begin{cases} \dfrac{\partial u}{\partial \beta} = m\dfrac{\partial u}{\partial \alpha}, & \dfrac{\partial u}{\partial \gamma} = n\dfrac{\partial u}{\partial \alpha}, \qquad \dfrac{\partial u}{\partial \delta} = l\dfrac{\partial u}{\partial \alpha} \\ \dfrac{\partial v}{\partial \beta} = m\dfrac{\partial v}{\partial \alpha}, & \dfrac{\partial v}{\partial \gamma} = n\dfrac{\partial v}{\partial \alpha}, \qquad \dfrac{\partial v}{\partial \delta} = l\dfrac{\partial v}{\partial \alpha} \end{cases}$$

Al sustituir los valores anteriores en la primera igualdad de (2), resulta:

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq + \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial \alpha}(d\alpha + md\,\beta + nd\,\gamma + ld\,\delta) = 0$$

o sea, recordando los valores (2') de *dz, dp, dq, dr, ds, dt*

$$\left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} + \alpha\frac{\partial u}{\partial r} + \beta\frac{\partial u}{\partial s} + \gamma\frac{\partial u}{\partial t}\right]dx + \left[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} + \beta\frac{\partial u}{\partial r} + \gamma\frac{\partial u}{\partial s} + \delta\frac{\partial u}{\partial t}\right]dy +$$

$$+\frac{\partial u}{\partial \alpha}(dz + md\,\beta + nd\,\gamma + ld\,\delta) = 0$$

Y según (3'), se obtiene:

$$d\,\alpha + md\,\beta + nd\,\gamma + ld\,\delta = -\frac{\left(\dfrac{\partial u}{dx}\right)dx + \left(\dfrac{\partial u}{dy}\right)dy}{\dfrac{\partial u}{\partial \alpha}}. \quad (5)$$

Si nos fijamos ahora en el primer miembro de esta última igualdad, sustituyendo en vez de $d\alpha, d\beta, d\gamma, d\delta$ sus valores respectivos (2), se tiene:

$$\alpha_1 dx + \beta_1(dy + mdx) + \gamma_1(ndx + mdy) + \delta_1(ldx + ndy) + l\varepsilon_1 dy + \frac{\left(\dfrac{\partial u}{\partial x}\right)dx + \left(\dfrac{\partial u}{\partial y}\right)dy}{\dfrac{\partial u}{\partial \alpha}} = 0;$$

ecuación de la misma forma que (1); luego cabe suponer

$$\frac{dx}{A} = \frac{dy + mdx}{B} = \frac{ndx + mdy}{C} = \frac{ldx + ndy}{D} = \frac{ldy}{E} = \frac{\left(\dfrac{\partial u}{\partial x}\right)dx + \left(\dfrac{\partial u}{\partial y}\right)dy}{P\dfrac{\partial u}{\partial \alpha}}$$

Y en virtud de (4'), después de simples reducciones,

$$\frac{dx}{A} = \frac{\dfrac{\partial u}{\partial \alpha}dy + \dfrac{\partial u}{\partial \beta}dx}{B\dfrac{\partial u}{\partial \alpha}} = \frac{\dfrac{\partial u}{\partial \gamma}dx + \dfrac{\partial u}{\partial \beta}dy}{C\dfrac{\partial u}{\partial \alpha}} = \frac{\dfrac{\partial u}{\partial \delta}dx + \dfrac{\partial u}{\partial \gamma}dy}{D\dfrac{\partial u}{\partial \alpha}} = \frac{\dfrac{\partial u}{\partial \delta}dy}{E\dfrac{\partial u}{\partial \alpha}} = \frac{\left(\dfrac{\partial u}{\partial x}\right)dx + \left(\dfrac{\partial u}{\partial y}\right)dy}{P\dfrac{\partial u}{\partial \alpha}}$$

De estas igualdades se deduce:

$$\frac{E}{A}\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial u}{\partial\delta}}=\frac{dy}{dx},\qquad \frac{B}{A}\frac{\partial u}{\partial\alpha}=\frac{\partial u}{\partial\beta}+\frac{\partial u}{\partial\alpha}\frac{E}{A}\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial u}{\partial\delta}},\qquad \frac{C}{A}\frac{\partial u}{\partial\alpha}=\frac{\partial u}{\partial\gamma}+\frac{\partial u}{\partial\beta}\frac{E}{A}\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial u}{\partial\delta}},$$

$$\frac{D}{A}\frac{\partial u}{\partial\alpha}=\frac{\partial u}{\partial\delta}+\frac{\partial u}{\partial\gamma}\frac{E}{A}\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial u}{\partial\delta}},\qquad \frac{P}{A}\frac{\partial u}{\partial\alpha}=\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)\frac{E}{A}\frac{\frac{\partial u}{\partial\alpha}}{\frac{\partial u}{\partial\delta}},$$

De donde, por fin, resultan las cuatro ecuaciones entre derivadas parciales que convienen a la ecuación dada (1), y que son las que a continuación se expresan:

$$(5')\qquad \begin{cases} E\left[\frac{\partial u}{\partial\alpha}\right]^2+A\frac{\partial u}{\partial\beta}\frac{\partial u}{\partial\delta}-B\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}=0 & (a)\\ E\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}+A\frac{\partial u}{\partial\gamma}\frac{\partial u}{\partial\delta}-C\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}=0, & (b)\\ E\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}+A\left[\frac{\partial u}{\partial\delta}\right]^2-D\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}=0, & (c)\\ E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}+A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}-P\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}=0, & (d)\end{cases}$$

A este punto nos interesa transformar el sistema anterior en otro de primer orden. Para ello multiplicaremos (*a*) por λ^2, (*b*) (*b*) por λ, y el resultado lo sumaremos con (*c*), de donde:

$$A\left[\frac{\partial u}{\partial\delta}\right]^2-\left(D+C\lambda+B\lambda^2\right)\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\alpha}+E\lambda\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}+E\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}+A\lambda\frac{\partial u}{\partial\gamma}\frac{\partial u}{\partial\delta}+E\lambda^2\left[\frac{\partial u}{\partial\alpha}\right]^2+A\lambda^2\frac{\partial u}{\partial\beta}\frac{\partial u}{\partial\delta}=0$$

Comparemos este resultado con la igualdad hipotética

$$(6)\quad \left[\lambda^2+\frac{\partial u}{\partial\alpha}+m\frac{\partial u}{\partial\beta}+n\frac{\partial u}{\partial\gamma}+l\frac{\partial u}{\partial\delta}\right]\left[E\frac{\partial u}{\partial\alpha}+m'\frac{\partial u}{\partial\delta}\right]=0$$

o sea, después de verificar operaciones, con

$$E\lambda^2\left[\frac{\partial u}{\partial\delta}\right]^2+Em\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}+En\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}+El\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}+m'\lambda^2\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}+mm'\frac{\partial u}{\partial\beta}\frac{\partial u}{\partial\delta}+m'n\frac{\partial u}{\partial\gamma}\frac{\partial u}{\partial\delta}+m'l\left[\frac{\partial u}{\partial\delta}\right]^2=0$$

y resulta:

$$E\lambda^2 = E\lambda^2, \qquad Em = E\lambda, \qquad En = E,$$

$$-El - m'\lambda^2 = D + C\lambda + B\lambda^2,$$

$$mm' = A\lambda^2 \qquad m'n = A\lambda, \qquad m'l = A$$

de donde,

$$m = \lambda, \qquad n = 1, \qquad m' = A\lambda, \qquad l = \frac{1}{\lambda},$$

$$(6') \quad A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

Si determinamos tres raíces de esta última ecuación de cuarto grado, tales como $\lambda_1, \lambda_2, \lambda_3$, y sustituimos los valores hallados en (6), puede establecerse el sistema de ecuaciones siguiente:

$$\left[\lambda_1^2 \frac{\partial u}{\partial \alpha} + \lambda_1 \frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} + \frac{1}{\gamma_1}\frac{\partial u}{\partial \delta}\right]\left[E\frac{\partial u}{\partial \alpha} + A\lambda_1 \frac{\partial u}{\partial \delta}\right] = 0,$$

$$\left[\lambda_2^2 \frac{\partial u}{\partial \alpha} + \lambda_2 \frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} + \frac{1}{\gamma_2}\frac{\partial u}{\partial \delta}\right]\left[E\frac{\partial u}{\partial \alpha} + A\lambda_2 \frac{\partial u}{\partial \delta}\right] = 0,$$

$$\left[\lambda_3^2 \frac{\partial u}{\partial \alpha} + \lambda_3 \frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} + \frac{1}{\gamma_3}\frac{\partial u}{\partial \delta}\right]\left[E\frac{\partial u}{\partial \alpha} + A\lambda_3 \frac{\partial u}{\partial \delta}\right] = 0,$$

$$A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial \delta} + E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha} - P\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \delta} = 0.$$

Este sistema, equivalente a (5'), es de primer orden.

Sin embargo, aun cabe obtener un nuevo sistema más sencillo, al igualar a cero el segundo factor de la primera ecuación y el primer factor de la segunda y tercera ecuación, resultando en su virtud:

$$(7) \quad \begin{cases} E\dfrac{\partial u}{\partial \alpha} + A\lambda_1 \dfrac{\partial u}{\partial \delta} = 0, \\[2ex] \lambda_2^2 \dfrac{\partial u}{\partial \alpha} + \lambda_2 \dfrac{\partial u}{\partial \beta} + \dfrac{\partial u}{\partial \gamma} + \dfrac{1}{\gamma_2}\dfrac{\partial u}{\partial \delta} = 0, \\[2ex] \lambda_3^2 \dfrac{\partial u}{\partial \alpha} + \lambda_3 \dfrac{\partial u}{\partial \beta} + \dfrac{\partial u}{\partial \gamma} + \dfrac{1}{\gamma_3}\dfrac{\partial u}{\partial \delta} = 0, \\[2ex] A\left(\dfrac{\partial u}{\partial x}\right)\dfrac{\partial u}{\partial \delta} + E\left(\dfrac{\partial u}{\partial y}\right)\dfrac{\partial u}{\partial \alpha} - P\dfrac{\partial u}{\partial \alpha}\dfrac{\partial u}{\partial \delta} = 0. \end{cases}$$

Por fin, procuraremos reducir este sistema a una sola ecuación lineal entre derivadas parciales de primer orden, para recabar el sistema de ecuaciones diferenciales totales que deben dar una primera integral de la ecuación dada (1).

Fácilmente se comprende que el sistema anterior (7), se puede transformar en el siguiente:

$$
(7')\quad \begin{cases}
E\left(\dfrac{\partial u}{\partial x}\right)+\lambda_1 E\left(\dfrac{\partial u}{\partial y}\right)+P\gamma_1\dfrac{\partial u}{\partial \delta}=0, & (a')\\[2ex]
E\dfrac{\partial u}{\partial \alpha}+A\lambda_1\dfrac{\partial u}{\partial \delta}=0, & (b')\\[2ex]
\lambda_2^3\dfrac{\partial u}{\partial \alpha}+\lambda_2^2\dfrac{\partial u}{\partial \beta}+\lambda_2\dfrac{\partial u}{\partial \gamma}+\dfrac{\partial u}{\partial \delta}=0, & (c')\\[2ex]
\lambda_3^3\dfrac{\partial u}{\partial \alpha}+\lambda_3^2\dfrac{\partial u}{\partial \beta}+\lambda_3\dfrac{\partial u}{\partial \gamma}+\dfrac{\partial u}{\partial \delta}=0, & (d')
\end{cases}
$$

Al multiplicar (*b*') por *m*, (*c*') por *n* y (*d*') por *l*, sumando los resultados con (a'), desarrollando al propio tiempo los valores (3') de $\left(\dfrac{\partial u}{\partial x}\right)$ y $\left(\dfrac{\partial u}{\partial y}\right)$, se deduce:

$$
E\frac{\partial u}{\partial x}-\lambda_1 E\frac{\partial u}{\partial y}+(Ep-\lambda_1 Eq)\frac{\partial u}{\partial z}+(Er-\lambda_1 Es)\frac{\partial u}{\partial p}+(Es-\lambda_1 Et)\frac{\partial u}{\partial q}+(E\alpha-\lambda_1 E\beta)\frac{\partial u}{\partial r}+(E\beta-\lambda_1 E\gamma)\frac{\partial u}{\partial s}+
$$

$$
+(E\gamma-\lambda_1 E\delta)\frac{\partial u}{\partial t}+\left(mE+n\lambda_2^3 l+\lambda_3^3\right)\frac{\partial u}{\partial \alpha}+\left(n\lambda_2^2+l\lambda_3^2\right)\frac{\partial u}{\partial \beta}+(n\lambda_2+l\lambda_3)\frac{\partial u}{\partial \gamma}+(mA\gamma_1+n+l+P\gamma_1)\frac{\partial u}{\partial \delta}=0.
$$

De donde resulta inmediatamente por principios de cálculo integral, el sistema de ecuaciones diferenciales totales siguientes:

$$
(8)\quad \begin{cases}
\dfrac{dx}{E}=\dfrac{dy}{-\lambda_1 E}=\dfrac{dz}{Ep-\lambda_1 Eq}\\[2ex]
\quad=\dfrac{dp}{Er-\lambda_1 Es}=\dfrac{dq}{Es-\lambda_1 Et}=\dfrac{dr}{E\alpha-\lambda_1 E\beta}\\[2ex]
\quad=\dfrac{ds}{E\beta-\lambda_1 E\gamma}=\dfrac{dt}{E\gamma-\lambda_1 E\delta}=\dfrac{d\alpha}{mE+n\lambda_2^8+l\lambda_3^3}\\[2ex]
\quad=\dfrac{d\beta}{n\lambda_2^2+l\lambda_3^2}=\dfrac{d\gamma}{n\lambda_2+l\lambda_3}=\dfrac{d\delta}{mA\lambda_1+n+l+P\lambda_1}
\end{cases}
$$

De las primeras igualdades se deduce:

$$(8')\quad \begin{cases} dy+\lambda_1 dx=0, & \\ dz-pdx-qdy=0, & dp-rdx-sdy=0, \\ dq-sdx-tdy=0, & dr-\alpha dx-\beta dy=0, \\ ds-\beta dx-\gamma dy=0, & dt-\gamma dx-\delta dy=0. \end{cases}$$

Para deducir los valores de las últimas cuatro igualdades de (8), es preciso eliminar *m*, *n*, *l*, y para ello podemos deducir de (8) las igualdades siguientes:

$$dx\left(n\lambda_2^2+l\lambda_3^2\right)=Ed\,\beta,$$
$$dx\left(n\lambda_2+l\lambda_3\right)=Ed\,\gamma,$$

que dan,

$$n=\frac{E\left(\lambda_3 d\,\beta-\lambda_3^2 d\,\gamma\right)}{\lambda_2\lambda_3\left(\lambda_2+\lambda_3\right)dx},\qquad l=\frac{E\left(\lambda_2^2 d\,\gamma-\lambda_2 d\,\beta\right)}{\lambda_2\lambda_3\left(\lambda_2+\lambda_3\right)dx};$$

de donde,

$$\frac{dx}{E}=\frac{d\,\alpha}{mE+\lambda_3^2\frac{E\left(\lambda_3 d\,\beta-\lambda_3^2 d\,\gamma\right)}{\lambda_2\lambda_3\left(\lambda_2+\lambda_3\right)dx}+\lambda_3^3\frac{E\left(\lambda_2^2 d\,\gamma-\lambda_2 d\,\beta\right)}{\lambda_2\lambda_3\left(\lambda_2+\lambda_3\right)dx}}$$

Después de toda reducción esta igualdad se transforma en:

$$m=\frac{d\,\alpha-\left(\lambda_2+\lambda_3\right)d\,\beta+\lambda_2\lambda_3 d\,\gamma}{dx}.$$

Considerando, por último, la igualdad:

$$\frac{dx}{E}=\frac{d\,\delta}{mA\lambda_1+n+l+P\lambda_1},$$

después de sustituir los valores obtenidos de *m*, *n*, *l*, y de simplificar el resultado, se halla,

$$A\lambda_1\lambda_2\lambda_3\left[d\,\alpha-\left(\lambda_2-\lambda_3\right)d\,\beta+\lambda_2\lambda_3 d\,\lambda\right]-E\left[d\,\beta-\left(\lambda_2-\lambda_3\right)d\,\gamma+\lambda_2\lambda_3 d\,\delta\right]+P\lambda_1\lambda_2\lambda_3 dx=0 \quad (9)$$

Este resultado cabe reducirlo, pues según la ecuación (6'),

$$\lambda_1\lambda_2\lambda_3\lambda_4=\frac{E}{A}\qquad \lambda_1+\lambda_2+\lambda_3+\lambda_4=-\frac{B}{A}$$

de donde, sustituyendo valores en (9)

$$\frac{E}{\lambda_4}\left[d\,\alpha+\left(\frac{B}{A}+\lambda_1+\lambda_4\right)d\,\beta+\frac{E}{A\lambda_1\lambda_4}d\,\gamma\right]-E\left[d\,\beta+\left(\frac{B}{A}+\lambda_1+\lambda_4\right)d\,\gamma+\frac{E}{A\lambda_1\lambda_4}d\,\delta\right]+\frac{PE}{A\lambda_4}dx=0$$

o sea,

$$A\lambda_1\lambda_4(d\,\alpha-\lambda_4 d\,\beta)+\lambda_1\lambda_4[B+A(\lambda_1+\lambda_4)](d\,\beta-\lambda_4 d\,\gamma)+E(d\,\gamma-\lambda_4 d\,\delta)+P\lambda_1\lambda_4 dx=0$$

Combinando las cuatro raíces de dos en dos, pueden deducirse de la igualdad anterior otras cinco, que junto con las (8') permitan hallar dos integrales particulares, tales como $u = a$ y $v = b$, las cuales, al suponer una de ellas función arbitraria de la otra, tal como $u = f(v)$, tendremos según los principios ya expuestos una integral primitiva de la ecuación dada (1).

De esta suerte queda resuelto el primer caso de los dos que habíamos indicado en el principio del presente artículo.

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II

Vamos a pasar ahora al segundo caso, conservando las mismas notaciones del primero. Sea la ecuación siguiente:

$$A\alpha_1 + B\beta_1 + C\gamma_1 + D\delta_1 + E\varepsilon_1 + F\left(\alpha_1\gamma_1 + \beta_1^2\right) + G\left(\alpha_1\delta_1 - \gamma_1\beta_1\right) + H\left(\alpha_1\varepsilon_1 - \delta_1\beta_1\right) + L\left(\beta_1\delta_1 - \gamma_1^2\right) +$$
$$+ M\left(\beta_1\varepsilon_1 - \delta_1\gamma_1\right) + N\left(\gamma_1\varepsilon_1 - \delta_1^2\right) + P = 0$$

Si $u = f(v)$ es una primera integral de esta ecuación, siendo f una función arbitraria, y u, $\boldsymbol{v}$, funciones de $x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$; cada una satisfará al sistema de ecuaciones que sigue:

$$(2)\quad \begin{cases}
P\dfrac{\partial f}{\partial\delta}\dfrac{\partial f}{\partial\alpha} - A\left(\dfrac{\partial f}{\partial x}\right)\dfrac{\partial f}{\partial\delta} - E\left(\dfrac{\partial f}{\partial y}\right)\dfrac{\partial f}{\partial\alpha} + H\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right) = 0, \\[2ex]
-A\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\delta} + B\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} + H\dfrac{\partial f}{\partial\beta}\left(\dfrac{\partial f}{\partial y}\right) + H\left(\dfrac{\partial f}{\partial x}\right)\dfrac{\partial f}{\partial\alpha} - M\left(\dfrac{\partial f}{\partial y}\right)\dfrac{\partial f}{\partial\alpha} - E\left[\dfrac{\partial f}{\partial\alpha}\right]^2 = 0, \\[2ex]
-A\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\delta} + C\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} - E\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\alpha} - F\left(\dfrac{\partial f}{\partial x}\right)\dfrac{\partial f}{\partial\delta} + H\left(\dfrac{\partial f}{\partial y}\right)\dfrac{\partial f}{\partial\gamma} + H\left(\dfrac{\partial f}{\partial x}\right)\dfrac{\partial f}{\partial\beta} = 0, \\[2ex]
-A\left[\dfrac{\partial f}{\partial\delta}\right]^2 + D\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} - E\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\alpha} - G\left(\dfrac{\partial f}{\partial x}\right)\dfrac{\partial f}{\partial\delta} + H\dfrac{\partial f}{\partial\gamma}\left(\dfrac{\partial f}{\partial x}\right) + H\left(\dfrac{\partial f}{\partial y}\right)\dfrac{\partial f}{\partial\delta} = 0, \\[2ex]
-F\dfrac{\partial f}{\partial\delta} + H\dfrac{\partial f}{\partial\beta} - M\dfrac{\partial f}{\partial\alpha} = 0, \\[2ex]
H\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\beta} + F\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\beta} - N\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\alpha} - L\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} = 0, \\[2ex]
-G\dfrac{\partial f}{\partial\delta} + H\dfrac{\partial f}{\partial\gamma} - N\dfrac{\partial f}{\partial\alpha} = 0, \\[2ex]
-F\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\delta} - G\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} + H\left[\dfrac{\partial f}{\partial\beta}\right]^2 + H\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\gamma} - M\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\alpha} - N\left[\dfrac{\partial f}{\partial\alpha}\right]^2 = 0, \\[2ex]
-G\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\delta} + H\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\gamma} + H\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} - H\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} + L\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} - M\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\alpha} = 0, \\[2ex]
-F\left[\dfrac{\partial f}{\partial\delta}\right]^2 + G\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\delta} + H\left[\dfrac{\partial f}{\partial\gamma}\right]^2 + H\dfrac{\partial f}{\partial\beta}\dfrac{\partial f}{\partial\delta} - M\dfrac{\partial f}{\partial\alpha}\dfrac{\partial f}{\partial\delta} - N\dfrac{\partial f}{\partial\gamma}\dfrac{\partial f}{\partial\alpha} = 0,
\end{cases}$$

Para llegar a este resultado, empezaremos suponiendo que $u = f(v)$ sea una primera integral de (1), bajo las condiciones anteriormente supuestas. Así, pues, tendremos:

$$\left(\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial \alpha}\alpha_1+\frac{\partial u}{\partial \beta}\beta_1+\frac{\partial u}{\partial \gamma}\gamma_1+\frac{\partial u}{\partial \delta}\delta_1 = f(v)\left[\left(\frac{\partial v}{\partial x}\right)+\frac{\partial v}{\partial \alpha}\alpha_1+\frac{\partial v}{\partial \beta}\beta_1+\frac{\partial v}{\partial \gamma}\gamma_1+\frac{\partial v}{\partial \delta}\delta_1\right],$$

$$\left(\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial \alpha}\beta_1+\frac{\partial u}{\partial \beta}\gamma_1+\frac{\partial u}{\partial \gamma}\delta_1+\frac{\partial u}{\partial \delta}\varepsilon_1 = f'(v)\left[\left(\frac{\partial v}{\partial y}\right)+\frac{\partial v}{\partial \alpha}\beta_1+\frac{\partial v}{\partial \beta}\gamma_1+\frac{\partial v}{\partial \gamma}\delta_1+\frac{\partial v}{\partial \delta}\varepsilon_1\right];$$

de donde resulta,

$$\frac{\left(\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial \alpha}\alpha_1+\frac{\partial u}{\partial \beta}\beta_1+\frac{\partial u}{\partial \gamma}\gamma_1+\frac{\partial u}{\partial \delta}\delta_1}{\left(\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial \alpha}\beta_1+\frac{\partial u}{\partial \beta}\gamma_1+\frac{\partial u}{\partial \gamma}\delta_1+\frac{\partial u}{\partial \delta}\varepsilon_1}=\frac{\left(\frac{\partial v}{\partial x}\right)+\frac{\partial v}{\partial \alpha}\alpha_1+\frac{\partial v}{\partial \beta}\beta_1+\frac{\partial v}{\partial \gamma}\gamma_1+\frac{\partial v}{\partial \delta}\delta_1}{\left(\frac{\partial v}{\partial y}\right)+\frac{\partial v}{\partial \alpha}\beta_1+\frac{\partial v}{\partial \beta}\gamma_1+\frac{\partial v}{\partial \gamma}\delta_1+\frac{\partial v}{\partial \delta}\varepsilon_1}.$$

Por fin, después de toda reducción, se obtiene:

$$\begin{aligned}
&\left[\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha}\right]\alpha_1\\
&+\left[\left(\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial \beta}-\left(\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial \alpha}+\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \alpha}-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \beta}\right]\beta_1\\
&+\left[\left(\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial \gamma}-\left(\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial \beta}+\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \beta}-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \gamma}\right]\gamma_1\\
&+\left[\left(\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial \delta}-\left(\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial \gamma}+\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \gamma}-\left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial \delta}\right]\delta_1\\
&+\left[\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial \delta}-\left(\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial \delta}\right]\varepsilon_1+\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta}-\frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \beta}\right]\left[\alpha_1\gamma_1-\beta_1^2\right]\\
&+\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \gamma}-\frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \gamma}\right]\left[\alpha_1\delta_1-\gamma_1\beta_1\right]+\left[\frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \delta}-\frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \delta}\right]\left[\alpha_1\varepsilon_1-\delta_1\beta_1\right]\\
&+\left[\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \gamma}-\frac{\partial v}{\partial \beta}\frac{\partial u}{\partial \gamma}\right]\left[\beta_1\delta_1-\gamma_1^2\right]+\left[\frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \delta}-\frac{\partial v}{\partial \beta}\frac{\partial u}{\partial \delta}\right]\left[\beta_1\varepsilon_1-\gamma_1\delta_1\right]\\
&+\left[\frac{\partial u}{\partial \gamma}\frac{\partial v}{\partial \delta}-\frac{\partial v}{\partial \gamma}\frac{\partial u}{\partial \delta}\right]\left[\gamma_1\varepsilon_1-\delta_1^2\right]+\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right]=0
\end{aligned}$$

En el supuesto de que esta ecuación sea equivalente a (1), cabrá escribir:

$$
(3)\quad \begin{cases}
\dfrac{\partial u}{\partial \alpha}\left(\dfrac{\partial v}{\partial y}\right)-\left(\dfrac{\partial u}{\partial y}\right)\dfrac{\partial u}{\partial \alpha}=A\mu, & (a)\\[2ex]
\left(\dfrac{\partial v}{\partial y}\right)\dfrac{\partial u}{\partial \beta}-\left(\dfrac{\partial v}{\partial x}\right)\dfrac{\partial u}{\partial \alpha}+\left(\dfrac{\partial u}{\partial x}\right)\dfrac{\partial v}{\partial \alpha}-\left(\dfrac{\partial u}{\partial y}\right)\dfrac{\partial v}{\partial \beta}=B\mu, & (b)\\[2ex]
\left(\dfrac{\partial v}{\partial y}\right)\dfrac{\partial u}{\partial \gamma}-\left(\dfrac{\partial v}{\partial x}\right)\dfrac{\partial u}{\partial \beta}+\left(\dfrac{\partial u}{\partial x}\right)\dfrac{\partial v}{\partial \beta}-\left(\dfrac{\partial u}{\partial y}\right)\dfrac{\partial v}{\partial \gamma}=C\mu, & (c)\\[2ex]
\left(\dfrac{\partial v}{\partial y}\right)\dfrac{\partial u}{\partial \delta}-\left(\dfrac{\partial v}{\partial x}\right)\dfrac{\partial u}{\partial \gamma}+\left(\dfrac{\partial u}{\partial x}\right)\dfrac{\partial v}{\partial \gamma}-\left(\dfrac{\partial u}{\partial y}\right)\dfrac{\partial v}{\partial \delta}=D\mu, & (d)\\[2ex]
\left(\dfrac{\partial u}{\partial x}\right)\dfrac{\partial v}{\partial \delta}-\left(\dfrac{\partial v}{\partial x}\right)\dfrac{\partial u}{\partial \delta}=E\,\mu, & (e)\\[2ex]
\dfrac{\partial u}{\partial \alpha}\dfrac{\partial v}{\partial \beta}-\dfrac{\partial v}{\partial \alpha}\dfrac{\partial u}{\partial \beta}=F\,\mu, & (f)\\[2ex]
\dfrac{\partial u}{\partial \alpha}\dfrac{\partial v}{\partial \gamma}-\dfrac{\partial v}{\partial \alpha}\dfrac{\partial u}{\partial \gamma}=G\,\mu, & (g)\\[2ex]
\dfrac{\partial u}{\partial \alpha}\dfrac{\partial v}{\partial \delta}-\dfrac{\partial v}{\partial \alpha}\dfrac{\partial u}{\partial \delta}=H\,\mu, & (h)\\[2ex]
\dfrac{\partial u}{\partial \beta}\dfrac{\partial v}{\partial \gamma}-\dfrac{\partial v}{\partial \beta}\dfrac{\partial u}{\partial \gamma}=L\,\mu, & (l)\\[2ex]
\dfrac{\partial u}{\partial \beta}\dfrac{\partial v}{\partial \delta}-\dfrac{\partial v}{\partial \beta}\dfrac{\partial u}{\partial \delta}=M\,\mu, & (m)\\[2ex]
\dfrac{\partial u}{\partial \gamma}\dfrac{\partial v}{\partial \delta}-\dfrac{\partial v}{\partial \gamma}\dfrac{\partial u}{\partial \delta}=N\,\mu, & (n)\\[2ex]
\left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial v}{\partial y}\right)-\left(\dfrac{\partial v}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right)=P\,\mu, & (p)
\end{cases}
$$

De la combinación de estas doce igualdades resultan las ecuaciones (2) que tratamos de demostrar. En efecto, por ejemplo, al multiplicar (*f*) por $\dfrac{\partial u}{\partial \gamma}$, (*g*) por $\dfrac{\partial u}{\partial \beta}$, y (*l*) por $\dfrac{\partial u}{\partial \alpha}$; se deduce:

$$L\frac{\partial u}{\partial \alpha}-G\frac{\partial u}{\partial \beta}+F\frac{\partial u}{\partial \gamma}=0$$

De un modo análogo se obtiene:

$$\text{de } (f),\ (h) \text{ y } (m),\quad F\frac{\partial u}{\partial \delta}-H\frac{\partial u}{\partial \beta}+M\frac{\partial u}{\partial \alpha}=0;$$

$$\text{de } (g),\ (h) \text{ y } (n),\quad G\frac{\partial u}{\partial \delta}-H\frac{\partial u}{\partial \gamma}+N\frac{\partial u}{\partial \alpha}=0;$$

$$\text{de } (l),\ (m) \text{ y } (n),\quad L\frac{\partial u}{\partial \delta}-M\frac{\partial u}{\partial \gamma}+N\frac{\partial u}{\partial \beta}=0;$$

Por medio de combinaciones más complicadas pueden resultar nuevas ecuaciones del sistema que se busca. En efecto, multiplicando:

$$(a)\ \text{por}\ \left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial \beta},\ (e)\ \text{por}\ \left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha},\ (h)\ \text{por}\ -\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\ \text{y}\ (p)\ \text{por}\ \frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \delta},$$

después de sumar los resultados, se halla:

$$A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial \delta}+E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha}-H\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)-P\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \delta}=0$$

Por otra parte, si multiplicamos: (a) por $\frac{\partial u}{\partial \beta}\frac{\partial u}{\partial \delta}$, ($b$) por $-\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \delta}$, ($e$) por $\left[\frac{\partial u}{\partial \alpha}\right]^2$, ($m$) por $\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha}$, y ($h$) por $-\left\{\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial \alpha}+\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \beta}\right\}$; en este caso resulta:

$$A\frac{\partial u}{\partial \beta}\frac{\partial u}{\partial \delta}-B\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \delta}+E\left[\frac{\partial u}{\partial \alpha}\right]^2+M\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \alpha}-H\left\{\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial \alpha}+\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial \beta}\right\}=0$$

Así podríamos continuar obteniendo nuevas ecuaciones del sistema (2), pero como quiera que es difícil atinar las diferentes combinaciones que deben tenerse presentes para alcanzar el fin que se desea, es preferible seguir una marcha general para obtenerlas todas de una vez. Supongamos que una integral de la ecuación dada (1), sea

$$\mathrm{F}(x,y,z,p,q,r,s,t,\alpha,\beta,\gamma,\delta)=0$$

Según lo que precede cabrá escribir:

$$\left(\frac{\partial \mathrm{F}}{\partial x}\right)+\frac{\partial \mathrm{F}}{\partial \alpha}\alpha_1+\frac{\partial \mathrm{F}}{\partial \beta}\beta_1+\frac{\partial \mathrm{F}}{\partial \gamma}\gamma_1+\frac{\partial \mathrm{F}}{\partial \delta}\delta_1=0$$

$$\left(\frac{\partial \mathrm{F}}{\partial y}\right)+\frac{\partial \mathrm{F}}{\partial \alpha}\beta_1+\frac{\partial \mathrm{F}}{\partial \beta}\gamma_1+\frac{\partial \mathrm{F}}{\partial \gamma}\delta_1+\frac{\partial \mathrm{F}}{\partial \delta}\varepsilon_1=0$$

de donde

$$\alpha_1=-\frac{\left(\frac{\partial \mathrm{F}}{\partial x}\right)+\frac{\partial \mathrm{F}}{\partial \beta}\beta_1+\frac{\partial \mathrm{F}}{\partial \gamma}\gamma_1+\frac{\partial \mathrm{F}}{\partial \delta}\delta_1}{\frac{\partial \mathrm{F}}{\partial \alpha}}$$

$$\varepsilon_1=-\frac{\left(\frac{\partial \mathrm{F}}{\partial y}\right)+\frac{\partial \mathrm{F}}{\partial \alpha}\beta_1+\frac{\partial \mathrm{F}}{\partial \beta}\gamma_1+\frac{\partial \mathrm{F}}{\partial \gamma}\delta_1}{\frac{\partial \mathrm{F}}{\partial \delta}}$$

Al sustituir estos valores en la ecuación dada (1), quitando denominadores y ordenando el desarrollo, según potencias de $\beta_1, \gamma_1, \delta_1,$ resultan las combinaciones siguientes:

$$\beta_1, \gamma_1, \delta_1, \beta_1^2, \gamma_1^2, \delta_1^2, \beta_1\gamma_1, \beta_1\delta_1, y_1\delta_1$$

y además el término independiente. Así, pues, en totalidad tendremos el desarrollo que a continuación se expresa:

$$
\begin{aligned}
& P\frac{\partial F}{\partial\delta}\frac{\partial F}{\partial\alpha} - A\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta} - E\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\alpha} + H\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right) \\
& + \beta_1\left\{-A\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta} + B\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} + H\frac{\partial F}{\partial\beta}\left(\frac{\partial F}{\partial y}\right) + H\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\alpha} - M\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\alpha} - E\left[\frac{\partial F}{\partial x}\right]^2\right\} \\
& + \gamma_1\left\{-A\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta} + C\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} - E\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha} + F\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta} + H\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\gamma} + H\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\beta}\right\} \\
& + \delta_1\left\{-A\left[\frac{\partial F}{\partial\delta}\right]^2 + D\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} - E\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha} - G\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta} + H\frac{\partial F}{\partial\gamma}\left(\frac{\partial F}{\partial x}\right) + H\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\delta}\right\} \\
& + \beta_1^2\left\{-F\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} + H\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha} - M\left[\frac{\partial F}{\partial\alpha}\right]^2\right\} \\
& + \gamma_1^2\left\{H\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\beta} - F\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta} - N\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha} - L\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}\right\} \\
& + \delta_1^2\left\{-G\left[\frac{\partial F}{\partial\delta}\right]^2 + H\frac{\partial F}{\partial\delta}\frac{\partial F}{\partial\gamma} - N\frac{\partial F}{\partial\delta}\frac{\partial F}{\partial\alpha}\right\} \\
& + \beta_1\gamma_1\left\{-F\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta} - G\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} + H\left[\frac{\partial F}{\partial\delta}\right]^2 + H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\gamma} - M\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha} - N\left[\frac{\partial F}{\partial\alpha}\right]^2\right\} \\
& + \beta_1\delta_1\left\{-G\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta} + H\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\gamma} + H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} - H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} + L\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} - M\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha}\right\} \\
& + \gamma_1\delta_1\left\{-F\left[\frac{\partial F}{\partial\delta}\right]^2 - G\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta} + H\left[\frac{\partial F}{\partial\gamma}\right]^2 + H\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta} - M\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta} - N\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha}\right\} = 0
\end{aligned}
$$

Igualando a cero los diferentes coeficientes, se obtiene:

$$P\frac{\partial F}{\partial\delta}\frac{\partial F}{\partial\alpha}-A\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta}-E\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\alpha}+H\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right)=0,$$

$$-A\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta}+B\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}+H\frac{\partial F}{\partial\beta}\left(\frac{\partial F}{\partial y}\right)+H\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\alpha}-M\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\alpha}-E\left[\frac{\partial F}{\partial x}\right]^2=0,$$

$$-A\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta}+C\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}-E\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha}+F\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta}+H\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\gamma}+H\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\beta}=0,$$

$$-A\left[\frac{\partial F}{\partial\delta}\right]^2+D\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}-E\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha}-G\left(\frac{\partial F}{\partial x}\right)\frac{\partial F}{\partial\delta}+H\frac{\partial F}{\partial\gamma}\left(\frac{\partial F}{\partial x}\right)+H\left(\frac{\partial F}{\partial y}\right)\frac{\partial F}{\partial\delta}=0,$$

$$-F\frac{\partial F}{\partial\delta}+H\frac{\partial F}{\partial\beta}-M\frac{\partial F}{\partial\alpha}=0,$$

$$H\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\beta}-F\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta}-N\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha}-L\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}=0,$$

$$-G\frac{\partial F}{\partial\delta}+H\frac{\partial F}{\partial\gamma}-N\frac{\partial F}{\partial\alpha}=0,$$

$$-F\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta}-G\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}+H\left[\frac{\partial F}{\partial\delta}\right]^2+H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\gamma}-M\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\alpha}-N\left[\frac{\partial F}{\partial\alpha}\right]^2=0,$$

$$-G\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta}+H\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\gamma}+H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}-H\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}+L\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}-M\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha}=0,$$

$$-F\left[\frac{\partial F}{\partial\delta}\right]^2-G\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\delta}+H\left[\frac{\partial F}{\partial\gamma}\right]^2+H\frac{\partial F}{\partial\beta}\frac{\partial F}{\partial\delta}-M\frac{\partial F}{\partial\alpha}\frac{\partial F}{\partial\delta}-N\frac{\partial F}{\partial\gamma}\frac{\partial F}{\partial\alpha}=0.$$

Conviene advertir que F representa una integral de (1), lo mismo que *u*, empero como entre las funciones *u* y *v* existe en todas las derivadas completa simetría, de ahí se infiere que F, así puede referirse a *u* como a *v*, quedando así comprobado con toda generalidad el grupo de fórmulas (2).

Además, según el primer procedimiento hemos encontrado las igualdades siguientes:

$$.(4)\quad\left\{\begin{array}{ll} L\dfrac{\partial u}{\partial\alpha}-G\dfrac{\partial u}{\partial\beta}+F\dfrac{\partial u}{\partial\gamma}=0, & a' \\[2ex] F\dfrac{\partial u}{\partial\delta}-H\dfrac{\partial u}{\partial\beta}+M\dfrac{\partial u}{\partial\alpha}=0, & b' \\[2ex] G\dfrac{\partial u}{\partial\delta}-H\dfrac{\partial u}{\partial\gamma}+N\dfrac{\partial u}{\partial\alpha}=0, & c' \\[2ex] L\dfrac{\partial u}{\partial\delta}-M\dfrac{\partial u}{\partial\gamma}+N\dfrac{\partial u}{\partial\beta}=0, & d' \end{array}\right.$$

mientras que por el segundo solo hemos hallado las (*b*') y (*c*').

Esto depende de que las (*a*') y (*d*') son consecuencias de (*b*') y (*c*'). En efecto, se tiene de (*b*') y (*c*').

$$M\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}+F\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\gamma}-H\frac{\partial u}{\partial\beta}\frac{\partial u}{\partial\gamma}=0,$$

$$G\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\beta}-H\frac{\partial u}{\partial\gamma}\frac{\partial u}{\partial\beta}+N\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}=0,$$

de donde, restando y teniendo presentes los valores (3) de M y N,

$$G\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\beta}-F\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\gamma}=M\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}-N\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}=\frac{1}{\mu}\left\{\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\gamma}\left[\frac{\partial u}{\partial\beta}\frac{\partial v}{\partial\delta}-\frac{\partial v}{\partial\beta}\frac{\partial u}{\partial\delta}\right]-\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\beta}\left[\frac{\partial u}{\partial\gamma}\frac{\partial v}{\partial\delta}-\frac{\partial v}{\partial\gamma}\frac{\partial u}{\partial\delta}\right]\right\}=$$

$$=\frac{1}{\mu}\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}\left[\frac{\partial u}{\partial\beta}\frac{\partial v}{\partial\gamma}-\frac{\partial u}{\partial\gamma}\frac{\partial v}{\partial\beta}\right]=L\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}$$

Así, pues, inmediatamente resultan las dos igualdades:

$$L\frac{\partial u}{\partial\alpha}-G\frac{\partial u}{\partial\beta}+F\frac{\partial u}{\partial\gamma}=0,\qquad L\frac{\partial u}{\partial\delta}-M\frac{\partial u}{\partial\gamma}+N\frac{\partial u}{\partial\beta}=0;$$

que corresponden con (*a*') y (*d*').

Con lo que precede queda completamente probado que la ecuación dada (1) depende del sistema general (2), y en este concepto solo falta demostrar que dicho sistema a la par va a depender de una ecuación entre derivadas parciales de primer orden y lineal, con lo cual podremos dar por resuelto el problema.

Para ello tomaremos del sistema general, las tres igualdades:

$$F\frac{\partial u}{\partial\delta}-H\frac{\partial u}{\partial\beta}+M\frac{\partial u}{\partial\alpha}=0,$$

$$A\frac{\partial u}{\partial\beta}\frac{\partial u}{\partial\delta}-B\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}-H\frac{\partial u}{\partial\beta}\left(\frac{\partial u}{\partial y}\right)-H\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\alpha}+M\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}+E\left[\frac{\partial u}{\partial\alpha}\right]^2=0,$$

$$A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}-H\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)-P\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\alpha}=0$$

De la primera se deduce:

$$\frac{\partial u}{\partial\beta}=\frac{1}{H}\left[F\frac{\partial u}{\partial\delta}+M\frac{\partial u}{\partial\alpha}\right]$$

al sustituir este valor en la segunda, se tiene:

$$\frac{A}{H}\left[F\,\frac{\partial u}{\partial\delta}+M\,\frac{\partial u}{\partial\alpha}\right]\frac{\partial u}{\partial\delta}-B\,\frac{\partial u}{\partial\alpha}\,\frac{\partial u}{\partial\delta}-\left(\frac{\partial u}{\partial y}\right)\left[F\,\frac{\partial u}{\partial\delta}+M\,\frac{\partial u}{\partial\alpha}\right]-H\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\alpha}+M\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}+E\left[\frac{\partial u}{\partial\alpha}\right]^2=0$$

y al multiplicar esta ecuación por la indeterminada λ, y sumando el resultado con la tercera, se halla por fin:

$$\frac{AF}{H}\lambda\left[\frac{\partial u}{\partial\delta}\right]^2+E\,\lambda\left[\frac{\partial u}{\partial\alpha}\right]^2+\frac{(AM-BH)\lambda-PH}{H}\,\frac{\partial u}{\partial\alpha}\,\frac{\partial u}{\partial\delta}-F\,\lambda\,\frac{\partial u}{\partial\delta}\left(\frac{\partial u}{\partial y}\right)-H\,\gamma\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\alpha}+$$

$$+A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}-H\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)=0$$

Consideremos ahora el producto

$$(5)\qquad \left[E\,\frac{\partial u}{\partial\alpha}+m\left(\frac{\partial u}{\partial x}\right)+n\,\frac{\partial u}{\partial\delta}\right]\left[\lambda\,\frac{\partial u}{\partial\alpha}+m'\left(\frac{\partial u}{\partial y}\right)+n'\,\frac{\partial u}{\partial\delta}\right]=0$$

o sea

$$E\,\lambda\left[\frac{\partial u}{\partial\alpha}\right]^2+m\,\lambda\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\alpha}+n\,\gamma\,\frac{\partial u}{\partial\delta}\,\frac{\partial u}{\partial\alpha}+Em'\,\frac{\partial u}{\partial\alpha}\left(\frac{\partial u}{\partial y}\right)+mm'\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+m'n\,\frac{\partial u}{\partial\delta}\left(\frac{\partial u}{\partial y}\right)$$

$$+En'\,\frac{\partial u}{\partial\alpha}\,\frac{\partial u}{\partial\delta}+mn'\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+nn'\left[\frac{\partial u}{\partial\delta}\right]^2=0$$

Al comparar este resultado con el anterior, se encuentra

$$E\,\lambda=E\,\lambda,\qquad m\,\lambda=-H\,\lambda,\qquad n\,\lambda+En'=\frac{(AM-BH)\lambda-PH}{H},$$

$$E'm=E,\qquad mm'=-H,\qquad nm'=-F\,\lambda,\qquad mn'=A,\qquad nn'=\frac{AF}{H}\lambda$$

lo que da:

$$m=-H,\qquad m'=1,\qquad n=-F\,\lambda,\qquad n'=-\frac{A}{H},$$

$$F\,\lambda^2-E\,\frac{A}{H}=\frac{(AM-BH)\lambda-PH}{H},\qquad FH\,\lambda^2+(AM-BH)\lambda+(EA-PH)=0$$

Por fin, si se sustituyen estos valores en (5), se tiene:

$$\left[E\,\frac{\partial u}{\partial\alpha}-H\left(\frac{\partial u}{\partial x}\right)-F\,\lambda\,\frac{\partial u}{\partial\delta}\right]\left[\lambda\,\frac{\partial u}{\partial\alpha}+\left(\frac{\partial u}{\partial y}\right)-\frac{A}{H}\,\frac{\partial u}{\partial\delta}\right]=0$$

Esta ecuación forma parte del nuevo sistema equivalente al anterior, con la ventaja de poderla descomponer en dos de primer orden.

Consideremos ahora las ecuaciones siguientes del sistema (2).

$$G\frac{\partial u}{\partial\delta}-H\frac{\partial u}{\partial\gamma}+N\frac{\partial u}{\partial\alpha}=0,$$

$$A\left[\frac{\partial u}{\partial\delta}\right]^2-D\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}+E\frac{\partial u}{\partial\gamma}\frac{\partial u}{\partial\alpha}+G\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}-H\frac{\partial u}{\partial\gamma}\left(\frac{\partial u}{\partial x}\right)-H\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\delta}=0,$$

$$A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}-H\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)-P\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\alpha}=0$$

De estas ecuaciones se originan como antes los cálculos que a continuación se expresan:

$$\frac{\partial u}{\partial\gamma}=\frac{1}{H}\left[G\frac{\partial u}{\partial\delta}+N\frac{\partial u}{\partial\alpha}\right];$$

$$A\left[\frac{\partial u}{\partial\delta}\right]^2-D\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}+\frac{E}{H}\left[G\frac{\partial u}{\partial\delta}+N\frac{\partial u}{\partial\alpha}\right]\frac{\partial u}{\partial\alpha}+G\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}-\left[G\frac{\partial u}{\partial\delta}+N\frac{\partial u}{\partial\alpha}\right]\left(\frac{\partial u}{\partial x}\right)-H\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\delta}=0$$

$$\frac{EN}{H}\mu\left[\frac{\partial u}{\partial\alpha}\right]^2+A\mu\left[\frac{\partial u}{\partial\delta}\right]^2-\frac{(DH-EG)\mu+PH}{H}\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}-N\mu\frac{\partial u}{\partial\alpha}\left(\frac{\partial u}{\partial x}\right)-H\mu\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+A\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial\delta}+$$

$$+E\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}-H\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)=0$$

Si comparamos esta ecuación con

$$(6)\quad\left[A\frac{\partial u}{\partial\delta}+m\left(\frac{\partial u}{\partial y}\right)+n\frac{\partial u}{\partial\alpha}\right]\left[\mu\frac{\partial u}{\partial\delta}+m'\left(\frac{\partial u}{\partial x}\right)+n'\frac{\partial u}{\partial\alpha}\right]=0,$$

o sea

$$A\mu\left[\frac{\partial u}{\partial\delta}\right]^2+m\mu\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\delta}+n\mu\frac{\partial u}{\partial\alpha}\frac{\partial u}{\partial\delta}+Am'\frac{\partial u}{\partial\delta}\left(\frac{\partial u}{\partial x}\right)+mm'\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)+m'n\frac{\partial u}{\partial\alpha}\left(\frac{\partial u}{\partial x}\right)+$$

$$An'\frac{\partial u}{\partial\delta}\frac{\partial u}{\partial\alpha}+mn'\left(\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial\alpha}+n'n\left[\frac{\partial u}{\partial\alpha}\right]^2=0$$

se deducen inmediatamente las igualdades

$$A\mu=A\mu,\qquad m\mu=-H\mu,\qquad n\mu+An'=-\frac{(DH-EG)\mu+PH}{H},$$

$$Am'=A,\qquad mm'=-H,\qquad m'n=-N\mu,\qquad mn'=E,\qquad nn'=\frac{EN}{H}\mu$$

lo que da:

$$m = -H, \qquad m' = 1, \qquad n = -N\mu, \qquad n' = -\frac{E}{H}, \qquad -N\mu^2 - \frac{AE}{H} = \frac{EG\mu - DH\mu - PH}{H},$$

$$NH\mu^2 + (EG - DG)\mu - (PH - AE) = 0.$$

Al sustituir estos valores en (6), se obtiene

$$\left[A\frac{\partial u}{\partial \delta} - H\left(\frac{\partial u}{\partial y}\right) - N\mu\frac{\partial u}{\partial \alpha}\right]\left[\mu\frac{\partial u}{\partial \delta} + \left(\frac{\partial u}{\partial x}\right) - \frac{E}{H}\frac{\partial u}{\partial \alpha}\right] = 0.$$

Esta ecuación forma parte del nuevo sistema equivalente al anterior (2); y puede también descomponerse en dos de primer orden, como en el caso precedente.

En este concepto, si tomamos los primeros factores de las dos ecuaciones halladas, junto con dos del primer sistema, tendremos un nuevo sistema que convendrá a la ecuación primitiva (1), permitiéndonos este último sistema pasar a una ecuación entre derivadas parciales de primer orden, siendo además lineal. En efecto, sea dicho sistema:

$$F\frac{\partial u}{\partial \delta} - H\frac{\partial u}{\partial \beta} + M\frac{\partial u}{\partial \alpha} = 0, \qquad (a'')$$

$$G\frac{\partial u}{\partial \delta} - H\frac{\partial u}{\partial \gamma} + N\frac{\partial u}{\partial \alpha} = 0, \qquad (b'')$$

$$E\frac{\partial u}{\partial \alpha} - H\left(\frac{\partial u}{\partial x}\right) - F\lambda\frac{\partial u}{\partial \delta} = 0, \qquad (c'')$$

$$A\frac{\partial u}{\partial \delta} - H\left(\frac{\partial u}{\partial y}\right) - N\mu\frac{\partial u}{\partial \alpha} = 0. \qquad (d'')$$

Si multiplicamos las igualdades (*b*''), (*c*'') y (*d* ''), respectivamente por *l*, *m*, *n*, y las sumamos con (*a*''), resulta:

$$nA\frac{\partial u}{\partial y} - Hn\left\{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} + \beta\frac{\partial u}{\partial r} + \gamma\frac{\partial u}{\partial s} + \delta\frac{\partial u}{\partial t}\right\} +$$

$$+ mE\frac{\partial u}{\partial \alpha} - Hm\left\{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} + \alpha\frac{\partial u}{\partial r} + \beta\frac{\partial u}{\partial s} + \gamma\frac{\partial u}{\partial t}\right\} -$$

$$- Nn\mu\frac{\partial u}{\partial \alpha} - Fm\lambda\frac{\partial u}{\partial \delta} - lG\frac{\partial u}{\partial \delta} - Hl\frac{\partial u}{\partial \gamma} + Nl\frac{\partial u}{\partial \alpha} + F\frac{\partial u}{\partial \delta} - H\frac{\partial u}{\partial \beta} + M\frac{\partial u}{\partial \alpha} = 0$$

o sea:

$$(nA - Fm\lambda + lG + F)\frac{\partial u}{\partial \delta} + (mE - Nn\mu + Nl + M)\frac{\partial u}{\partial \alpha} - Hn\frac{\partial u}{\partial y} - Hm\frac{\partial u}{\partial x} - (Hnq + Hmp)\frac{\partial u}{\partial z} -$$

$$-(Hns + Hmp)\frac{\partial u}{\partial p} - (Hnt + Hms)\frac{\partial u}{\partial q} - (Hn\beta + Hm\alpha)\frac{\partial u}{\partial r} - (Hn\gamma + Hm\beta)\frac{\partial u}{\partial s} - (Hn\delta + Hm\lambda)\frac{\partial u}{\partial t} - H\frac{\partial u}{\partial \beta} - Hl\frac{\partial u}{\partial \gamma} = 0.$$

Habiendo llegado a una ecuación entre derivadas parciales de primer orden y lineal, ya sabemos por los principios de cálculo integral, que se resuelve refiriéndola al sistema de ecuaciones diferenciales siguiente:

$$(7)\quad \left\{ \begin{aligned} & \frac{dx}{-Hm} = \frac{dy}{-Hn} = \frac{dz}{-(Hnq+Hmp)} = \frac{dp}{-(Hns+Hmp)} \\ & = \frac{dq}{-(Hnt+Hms)} = \frac{dr}{-(Hn\beta+Hm\alpha)} = \frac{ds}{-(Hn\gamma+Hm\beta)} \\ & = \frac{dt}{-(Hn\delta+Hm\gamma)} = \frac{d\alpha}{mE-Nn\mu+Nl+M} \\ & = \frac{d\beta}{-H} = \frac{d\gamma}{-Hl} = \frac{d\delta}{nA-Fm\lambda+lG+F} = \frac{d\beta}{-H} = \frac{d\gamma}{-Hl} = \frac{d\delta}{nA-Fm\lambda+lG+F} \end{aligned} \right.$$

Por fin, para eliminar *l*, *m*, *n*, bastará atender a los desarrollos siguientes que se deducen de las igualdades anteriores;

$$m = \frac{dx}{d\beta}, \qquad n = \frac{dy}{d\beta}, \qquad l = \frac{d\gamma}{d\beta}, \qquad -\frac{d\beta}{H} = \frac{d\alpha}{\frac{dx}{d\beta}E - N\frac{dy}{d\beta}\mu + N\frac{d\gamma}{d\beta} + M},$$

$$Edx - N\mu dy + Nd\gamma + Md\beta + Hd\alpha = 0, \qquad -\frac{d\beta}{H} = \frac{d\delta}{\frac{dy}{d\beta}A - F\frac{dx}{d\beta}\lambda + \frac{d\gamma}{d\beta}G + F},$$

$$Ady - F\lambda dx + Gd\gamma + Fd\beta + Hd\delta = 0$$

De esta suerte, uniendo estas dos últimas ecuaciones diferenciales totales ordinarias con todas las demás que se pueden deducir de las igualdades (7), junto con los diferentes valores de λ y μ que se deducen de las ecuaciones de segundo grado de que dependen, cabrá formar diferentes sistemas de ecuaciones diferenciales totales ordinarias; si por algunos de estos sistemas se pueden deducir dos integrales de la forma $u = a$, $v = b$; suponiendo que una sea función arbitraria de la otra, esto es, $u = f(v)$, tendremos con ello una primera integral de la ecuación dada, que es lo que nos habíamos propuesto determinar, en el caso general de una ecuación entre derivadas parciales de cuarto orden con dos variables independientes.

Zaragoza, años 1903 y 1904
Lauro Clariana Ricart